ON THE SRAFFA-LEONTIEF MODEL

by

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Abstract. This paper considers the matrix forms of the well-known Sraffa-Leontief income distribution model $px = (1+r)pAx + w^* p(I-A)x$. The equivalence between these matrix forms and the set of simpler models, including the Sraffian condition of linear relations between the rate of profits r and wage rate w^* will be explored. Further, the condition that the prices vector p and the commodities vector x are the left hand and the right hand eigenvectors of the matrix A of direct inputs and that these vectors are the fixed points of the Sraffian standard commodities-standard prices matrix will be evaluated.
The paper will then explore links between the Sraffa-Leontief system and the multiplier product matrix (MPM) for the matrix A to consider new insights generated through visualization with the help of the artificial economic landscape. Furthermore, the connections between MPM and the Sraffian standard commodities-standard prices matrix and their minimal information properties are proven.

Key words: Sraffa-Leontief income distribution model, Multiplier Product matrix, Sraffian standard commodities-standard prices matrix, minimal information properties.

1. Introduction

The contributions of Sraffa (1960) to the understanding of economic structure have been significantly advanced in recent years by the interpretative assessments of Steenge (1995, 1997). In this paper, these interpretations are complemented with some additional modifications that attempt to simplify the presentation of the Sraffa-Leontief system. In the next section, the standard Sraffian model is presented and some of the initial modifications are outlined. In section 3, the concept of the multiplier product matrix (mpm) is introduced but in a modified form; inside of considering the Leontief inverse matrix, the mpm methodology is applied to the matrix of direct coefficients to afford a direct link with the Sraffa system. In section 4, the minimum information properties of the mpm and Sraffian matrix are presented and are shown to be directly related in sections 5 and 6. Section 7 provides an empirical example from an input-output table for the Chicago metropolitan region. The paper concludes with some brief summary remarks and suggestions for further extensions.

2. The Sraffian model.

2.1 Sraffian Prices decomposition matrix Primary model and Commodities decomposition matrix Dual model.

The simplest and most obvious way to construct the Sraffian income distribution model is as follows (*cf.* Pasinetti, 1977): the input-output model is defined in the usual manner:

$$x = Ax + f \qquad (1)$$

where x is a gross output, A the matrix of input coefficients, and f is the final demand.

Introducing the vector of prices p we obtain:

$$px = pAx + pf\ , \qquad (2)$$

where conventionally assumed that the vector of direct labor coefficients

$$l = pf\ , \qquad (3)$$

and the total quantity of labor is $lx = L=1$.

Income distribution theory requires:

$$pf = rpAx + w^* pf \qquad (4)$$

where r is the uniform rate of profits and w^* is the wage rate .measured as a share of the national income, so $0 \le w^* \le 1$.

Substituting (3) and (4) into (2), the Sraffian income distribution model may be obtained:

$$px = (1+r)pAx + w^* p(I-A)x \qquad (5)$$

In matrix form, this model can be presented in the form of a prices decomposition matrix primal model:

$$p = (1+r)pA + w^* p(I-A) \qquad (6)$$

and in the form of the commodities decomposition matrix dual model:

$$x = (1+r)Ax + w^*(I-A)x \qquad (7)$$

Both matrix models will yield model (5) by post-multiplication of (6) on *x* and pre-multiplication of (7) on *p*.

2.2. Sraffian matrix and decomposition of the matrix A of direct inputs.

Consider a non-negative matrix, *A*, of direct input coefficients (in the following, it is not necessary to adopt the usual assumption of decomposability and primitivity of matrix *A*). From the theory of non-negative matrices (see, for example, Horn and Johnson, 1985, p. 503), it follows that there are non-negative eigenvectors (left hand and right hand) *p** and *x** of *A* corresponding to a non-negative eigenvalue μ .

$$p^* A = \mu p^*, \quad Ax^* = \mu x^* \tag{8}$$

such that μ is a simple eigenvalue,

It is well known that if the sum of the elements of each column of the matrix of direct inputs is smaller than 1 and, of course, greater than 0, then the maximal eigenvalue μ is in the interval $0 < \mu < 1$. This follows from the well-known inequality for the spectral radius $\rho(A)$of eigenvalues of positive matrix *A* (see, for example, Horn and Johnson, 1985, p.346):

$$\min_i \sum_j a_{i\ j} \ \le \ \rho(A) \le \max_i \sum_j a_{i\ j} \tag{9}$$

The simplicity of μ means that the eigenspaces of all left hand and right hand eigenvectors, corresponding to μ , are one-dimensional spaces; i.e., each left hand eigenvector is proportional to *p**, and each right hand eigenvector is proportional to *x**.

Let us assume (following Steenge, 1995, p. 57) that

$$p^* x^* = 1 \tag{10}$$

and consider the matrix $S = x^* p^*$. This matrix will assume a major role in the subsequent analysis and it will be referred to as the *Sraffian matrix* with the vector *x**, the vector of standard commodities, and the vector *p**, the vector of standard prices. Note that this definition of standard commodities is different from the conventional Sraffa definition: Sraffa labeled as standard commodities the vector of final demand *f** generating the gross output *x**:

$$f^* = x^* - Ax^* = (1-\mu)x^*.$$

For standardization we will assume the constant value of the Sraffian standard commodity

$$pf^* = p^* f = p^* f^* = (1-\mu) \tag{11}$$

Condition (10) means that these vectors are fixed points of the transformation *S:*

$$p^*S = p^*, \quad Sx^* = x^* \tag{12}$$

The conditions (8) imply that

$$AS = SA = \mu S \tag{13}$$

Consider further *C*=*I* - *S*. Obviously

$$\begin{aligned} &S^2 = S, C^2 = C, \quad CS = SC = 0, \\ &AS = SA = \mu S, \; AC = CA \end{aligned} \tag{14}$$

i.e.,

$$A = A(S+C) = \mu S + CA \tag{15}$$

Goodwin (1983) and Steenge (1995) used in their considerations the complicated fine structure (Perronean properties) of the spectrum and spectral decomposition of the matrices with non-negative components. Alternatively, use will be made here only of the decomposition (15), where the Perronean eigenvector, corresponding to the larger eigenvalue, and the one-dimensional eigenspace that splits from the eigenvalues structure. This splitting essentially will simplify the proofs.

2.3. Sraffa's linear Wage-Profit trade-off.

If in the Sraffian model (5), an arbitrary vector of prices *p* and gross output equal to *x**, are chosen, then substituting gross output *x* by the eigenvector *x** in the Sraffa-Leontief model (5), using (8) and dividing by *px** one obtains:

$$1=(1+r)\mu+w^*(1-\mu) \tag{16}$$

that implies the Sraffa linear relation between the rate of profits *r* and the wage rate *w** (see Sraffa, 1960, p.22, see also Pasinetti, 1977, p. 115):

$$r=(1-w^*)\frac{1-\mu}{\mu} \tag{17}$$

Obviously, the maximal rate of profits, corresponding to the case $w^* = 0$ is equal to

$$r_{\max}=\frac{1-\mu}{\mu} \tag{18}$$

The same situation will occurs when the choice is made of an arbitrary gross output *x* and the vector of prices *p**.

2.4. Equivalence theorems for the Primary Sraffa-Leontief matrix model.

The linear wage-profit trade-off (equations 17 - 18) implies the following statements:

Theorem P1. *If* (α) *in the Primary Sraffa-Leontief matrix model* (6) $w^*\neq 1$, *then* (β) *in this model is equivalent to*

$$\begin{cases} 1-w^*=\mu(1+r-w^*) \\ \mu p = pA \end{cases} \tag{19}$$

Proof: $(\alpha)\Rightarrow(\beta)$

The model (6) can be rewritten in the following form:

$$(1-w^*)\,p=(1+r-w^*)\,pA \tag{20}$$

and, as shown above in section *2.3* implies (17 – 18). The conditions (17 – 18) mean that

$$1 - w^* = \mu(1 + r - w^*) \tag{21}$$

and this implies, after substitution in (20), that:

$$\mu p = pA \tag{22}$$

$(\beta) \Rightarrow (\alpha)$

Contrarily, if the conditions (19) are true, then multiplying (22) by *1+r - w** and using (21) one obtains (20), which concludes the proof of equivalence.

2.5. The Steenge equivalence condition.

The following proposition is the reformulation and clarification of the considerations of Steenge, (1995, pp.63-66). Steenge proved that in Propositions P1 and D1 both conditions in (β) are equivalent in the Sraffa model. In his proof, he used the fine structure of the all eigenvalues and eigenvectors of positive matrices.

Proposition P1. *If* (α) *in the Primary Sraffa-Leontief matrix model* (7) $w^* \neq 1$, *then* (β) *this model is equivalent to*

$$\begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ p = pS \end{cases} \tag{23}$$

Proof: $(\alpha) \Rightarrow (\beta)$

If condition (22) is true then

$$\mu pC = \mu p(I - S) = \mu p - \mu pS = pA - \mu pS = p(A - \mu S) = pAC$$

This condition means that the vector *pC* is a left-hand eigenvector for *A*, and because the eigenspace corresponding to the simple eigenvalue μ is one-dimensional, then the vector *pC* is proportional to *p**, i.e., there is a number λ such that $pC = \lambda p^*$. Using (9) one obtains $pC = \lambda p^* S$. Therefore $pC = pC^2 = (\lambda p^* S)C = 0$. Thus,

$$p = pS + pC = pS \tag{24}$$

$(\beta) \Rightarrow (\alpha)$

Conversely, if $p = pS$, then $pC = 0$, and $pCA = pAC = 0$. Therefore

$$pA = p(\mu S + AC) = \mu pS = \mu p$$

Thus, (23) is equivalent to (19).

2.6. Equivalence theorems for the Dual Sraffa-Leontief matrix model.

Analogously, the following statement can be proven for the Dual Sraffa-Leontief model (7).

Theorem D1. *If* (α) *in the Primary Sraffa-Leontief matrix model* (7) $w^* \neq 1$, *then* (β) *in this model is equivalent to*

$$\begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ \mu x = Ax \end{cases} \tag{25}$$

Proposition D1. (Steenge, 1995). *If* (α) *in the Dual Sraffa-Leontief matrix model* (7) $w^* \neq 1$, *then* (β) *this model is equivalent to*

$$\begin{cases} 1 - w^* = \mu(1 + r - w^*) \\ x = Sx \end{cases} \tag{26}$$

3. The Multiplier Product Matrix (MPM).

3.1 The definition of MPM.

In this section, a connection between the Sraffa standard commodity system and the multiplier product matrix will be revealed. The definition of the direct inputs multiplier product matrix (MPM) is as follows: let $A = \left[a_{ij}\right]$ be a matrix of direct inputs in the input-output system, and let $m_{\bullet j}$ and $m_{i \bullet}$ be the column and row sums of this matrix. Following Chenery and Watanabe (1958) these are defined as:

$$m_{\bullet j} = \sum_{i=1}^{n} a_{ij}, \quad m_{i \bullet} = \sum_{j=1}^{n} a_{ij} \tag{27}$$

Let V be the global intensity of the matrix A:

$$V = V(A) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \tag{28}$$

Then, the input-output multiplier product matrix (MPM) is defined as:

$$M = M(A) = \frac{1}{V}\left\| m_{i\bullet} m_{\bullet j} \right\| = \frac{1}{V}\begin{pmatrix} m_{1\bullet} \\ m_{2\bullet} \\ \vdots \\ m_{n\bullet} \end{pmatrix}\begin{pmatrix} m_{\bullet 1} & m_{\bullet 2} & \cdots & m_{\bullet n} \end{pmatrix} = \left[m_{ij} \right] \tag{29}$$

Introducing the vectors of column and row sums:

$$M_r(A) = \begin{pmatrix} m_{1\bullet} \\ m_{2\bullet} \\ \vdots \\ m_{n\bullet} \end{pmatrix}; \; M_c(A) = \begin{pmatrix} m_{\bullet 1} & m_{\bullet 2} & \cdots & m_{\bullet n} \end{pmatrix} \tag{30}$$

one obtains the following expression of MPM matrix:

$$M(A) = \frac{1}{V} M_r(A) M_c(A) \tag{31}$$

The properties of the MPM will now be considered in the context of the following issues: (i) the hierarchy of backward and forward direct inputs linkages and their economic landscape associated with the cross-structure of the MPM; and (ii) the minimum information properties of the MPM.

3.2. Economic Cross-Structure Landscapes of MPM and the Rank-Size Hierarchies of input backward and forward Linkages.

In this subsection, the main notions and results of the Rasmussen-Hirshman key sector analysis of backward and forward linkages for the direct and indirect inputs of Leontief inverse will be transferred to the case of direct inputs. To this end, the transformation in the classical theory of key sectors (*cf.* Sonis, Hewings and Guo, 1995, Sonis and Hewings, 1999) involves replacement of the Leontief inverse by the consideration of the matrix of direct inputs *A*. Following this analogy, and the ideas of Rasmussen (1956), two types of indices will be defined, drawing on entries in the matrix A of direct inputs:

1. Power of dispersion of direct inputs for the backward linkages, $DIBL_j$, as follows:

$$
\begin{aligned}
DIBL_j &= \frac{1}{n}\sum_{i=1}^{n} {}_{ij} \bigg/ \frac{1}{n^2}\sum_{i,j=1}^{n} a_{ij} = \\
&= \frac{1}{n} m_{\bullet j} \bigg/ \frac{1}{n^2} V = m_{\bullet j} \bigg/ \frac{1}{n} V
\end{aligned}
\tag{32}
$$

and

2. The indices of the sensitivity of dispersion of direct inputs for forward linkages, $DIFL_i$, as follows:

$$
\begin{aligned}
DIFL_i &= \frac{1}{n}\sum_{j=1}^{n} a_{ij} \bigg/ \frac{1}{n^2}\sum_{i,j=1}^{n} a_{ij} = \\
&= \frac{1}{n} m_{i\bullet} \bigg/ \frac{1}{n^2} V = m_{i\bullet} \bigg/ \frac{1}{n} V
\end{aligned}
\tag{33}
$$

A direct inputs key sector, K, is usually defined as one in which both indices are greater than 1

The definitions of backward and forward linkages provided by (32) and (33) imply that the rank-size hierarchies (rank-size ordering) of these indices coincide with the rank-size hierarchies of the column and row sums. It is important to underline, in this connection, that the column and row sums for MPM are the same as those for the matrix of direct inputs A:

$$
\begin{aligned}
\sum_{j=1}^{n} m_{ij} &= \frac{1}{V}\sum_{j=1}^{n} m_{i\bullet} m_{\bullet j} = m_{i\bullet} \\
\sum_{i=1}^{n} m_{ij} &= \frac{1}{V}\sum_{i=1}^{n} m_{i\bullet} m_{\bullet j} = m_{\bullet j}
\end{aligned}
\tag{34}
$$

Thus, the structure of the MPM is essentially connected with the properties of sectoral direct inputs backward and forward linkages.

The structure of the matrix, M, can be ascertained in the following fashion: consider the largest column sum, $m_{\bullet j}$ and the largest row sum, $m_{i\bullet}$ of the matrix A. Further, the element,

$m_{i_0 j_0} = \frac{1}{V} m_{i_0 \bullet} m_{\bullet j_0}$, is located in the place (i_0, j_0) of the matrix, *M*. Moreover, all rows of the matrix, *M*, are proportional to the i_0^{th} row, and the elements of this row are larger than the corresponding elements of all other rows. The same property applies to the j_0^{th} column of the same matrix. Hence, the element located in (i_0, j_0) defines the center of the largest *cross* within the matrix, *M*. If this cross is excluded from *M*, then the second largest cross can be identified and so on. Thus, the matrix, *M*, contains the rank-size sequence of crosses. One can reorganize the locations of rows and columns of *M* in such a way that the centers of the corresponding crosses appear on the main diagonal. In this fashion, the matrix will be reorganized in such a way that a descending *economic landscape* will be apparent (see figure 1).

<<insert figures 1, 2 here>>

This rearrangement also reveals the descending rank-size hierarchies of the indices for direct forward and backward linkages. Inspection of that part of the landscape with indices > 1 (the criterion for specification of direct inputs key sectors) will enable the identification of the key sectors (see figure 2). However, it is important to stress that the construction of the economic landscape for different regions or for the same region at different points in time would create the possibility for the establishment of taxonomy of these economies. Moreover, the superposition of the hierarchy of one region on the landscape of another region provides a clear visual representation of the similarities and differences in the linkage structure of these regions.

It is important to stress, as will be shown in section 6, that the Sraffian standard commodities-standard prices matrix *S* coincides with the multiplier product matrix *M*(*S*). Hence, the Sraffian matrix has the same cross-structure defined by rank-size hierarchies of components of vectors of standard commodities and standard prices (see figure 3).

<<insert figure 3 here>>

4. Minimum Information Properties of the MPM and *S*.

4.1. Definition of information of the positive matrix.

Consider all positive matrices, $\Psi = \left[\psi_{ij}\right]$ with the property that the row and column multipliers are equal to those of the matrix A:

$$\sum_j \psi_{ij} = m_{i\,\bullet}, \quad \sum_i \psi_{ij} = m_{\bullet\, j} \tag{35}$$

Obviously, $\sum_{i,j} \psi_{ij} = V$.

We can convert each positive matrix Ψ into the two-dimensional probabilistic distribution matrix, $P = \left[p(i,j)\right]$ with the components:

$$p(i,j) = \Psi_{ij} / V \tag{36}$$

Therefore, we can attribute to each positive matrix Ψ the Shannon information (*INF*):

$$INF\Psi = INFP = \sum_{i,j} p(i,j) \ln p(i,j) = \sum_{i,j} \frac{\psi_{ij}}{V} \ln \frac{\psi_{ij}}{V} \tag{36}$$

4.2. Minimum information of MPM and S.

Recall the well-known Shannon information inequality (Shannon and Weaver, 1964, p. 51):

$$\sum_{i,j} p(i,j) \ln p(i,j) \geq \sum_{i,j} p(i,j) \ln \sum_j p(i,j) + \sum_{i,j} p(i,j) \ln \sum_{i,j} p(i,j) \tag{37}$$

This implies that each positive matrix, Ψ, satisfying the condition (38) may be shown as:

$$\begin{aligned}
INF\Psi &= \sum_{i,j} \frac{\Psi_{ij}}{V} \ln \frac{\Psi_{ij}}{V} \geq \sum_{ij} \frac{\Psi_{ij}}{V} \ln \sum_j \frac{\Psi_{ij}}{V} + \sum_{ij} \frac{\Psi_{ij}}{V} \ln \sum_{ij} \frac{\Psi_{ij}}{V} = \\
&= \sum_{ij} \frac{\Psi_{ij}}{V} \ln \frac{m_{i\,\bullet}}{V} + \sum_{ij} \frac{\Psi_{ij}}{V} \ln \frac{m_{\bullet\, j}}{V} = \sum_i \left(\sum_j \frac{\Psi_{ij}}{V} \right) \ln \frac{m_{i\,\bullet}}{V} + \sum_j \left(\sum_i \frac{\Psi_{ij}}{V} \right) \ln \frac{m_{\bullet\, j}}{V} = \\
&= \sum_i \frac{m_{i\,\bullet}}{V} \ln \frac{m_{i\,\bullet}}{V} + \sum_j \frac{m_{\bullet\, j}}{V} \ln \frac{m_{\bullet\, j}}{V} = \sum_i \frac{m_{i\,\bullet}}{V^2} \left(\sum_j m_{\bullet\, j} \right) \ln \frac{m_{i\,\bullet}}{V} + \sum_j \frac{m_{\bullet\, j}}{V^2} \left(\sum_i m_{i\,\bullet} \right) \ln \frac{m_{\bullet\, j}}{V} = \\
&= \sum_{ij} \frac{m_{i\,\bullet} m_{\bullet\, j}}{V^2} \left(\ln \frac{m_{i\,\bullet}}{V} + \ln \frac{m_{\bullet\, j}}{V} \right) = \sum_{ij} \frac{m_{i\,\bullet} m_{\bullet\, j}}{V^2} \ln \frac{m_{i\,\bullet} m_{\bullet\, j}}{V^2} = INFM.
\end{aligned} \tag{38}$$

Then:

$$INF\Psi \geq INFM \tag{39}$$

and the multiplier product matrix, *M*, has a minimal information property (Sonis, 1968).

The matrix *M* may be considered to represent the most homogeneous distribution of the components of the column and row sums of the matrix *A*. A further perspective may be offered; in the case of equal column and row sums, the economic landscape will be a flat, horizontal plane.

The MPM depends on the column and row sums only and, thus, represents only the aggregate characteristics of the direct interactions of each sector with the rest of the economy. Thus, MPM does not take into account the specifics of the pair-wise sectoral interactions between direct inputs; MPM can be considered as an aggregate representation of some sector equalization tendency in the economic interaction between sectors. Of course the same property of minimal information hold for the Sraffian matrix.

5. Properties of column and row multipliers of the product of positive matrices

In the following, this important property of the multipliers of the product of two positive matrices will be used. Consider the product $A = A_1 A_2 = \left[a_{ij}\right]$ of two matrices $A_1 = \left[a_{ij}^1\right]$, $A_2 = \left[a_{ij}^2\right]$. Let

$$\begin{aligned} m_{\bullet j} &= \sum_{i=1}^{n} a_{ij}, \quad m_{i \bullet} = \sum_{j=1}^{n} a_{ij} \\ m_{\bullet j}^1 &= \sum_{i=1}^{n} a_{ij}^1, \quad m_{i \bullet}^1 = \sum_{j=1}^{n} a_{ij}^1 \\ m_{\bullet j}^2 &= \sum_{i=1}^{n} a_{ij}^2, \quad m_{i \bullet}^2 = \sum_{j=1}^{n} a_{ij}^2 \end{aligned} \tag{40}$$

be the column and row multipliers of these matrices. Let $V = \sum_{ij} a_{ij}$ be the global intensity of the matrix *A*. Further, specify the following vectors of column and row multipliers:

$$M_c(A) = [m_{\bullet 1} m_{\bullet 2} \ldots m_{\bullet n}];\ M_c(A_1) = \left[m_{\bullet 1}^1 m_{\bullet 2}^1 \ldots m_{\bullet n}^1\right];\ M_c(A_2) = \left[m_{\bullet 1}^2 m_{\bullet 2}^2 \ldots m_{\bullet n}^2\right]$$
$$M_r(A) = \begin{bmatrix} m_{1\bullet} \\ m_{2\bullet} \\ \vdots \\ m_{n\bullet} \end{bmatrix};\ M_r(A_1) = \begin{bmatrix} m_{1\bullet}^1 \\ m_{2\bullet}^1 \\ \vdots \\ m_{n\bullet}^1 \end{bmatrix};\ M_r(A_2) = \begin{bmatrix} m_{1\bullet}^2 \\ m_{2\bullet}^2 \\ \vdots \\ m_{n\bullet}^2 \end{bmatrix} \tag{41}$$

The following formulae can be checked by direct calculations of the components of corresponding vectors and matrices:

$$\begin{aligned} M_c(A) &= M_c(A_1) A_2; \\ M_r(A) &= A_1 M_r(A_2); \\ V(A) &= M_c(A_1) M_r(A_2) \end{aligned} \tag{42}$$

6. Interconnections between MPM and *S*.

It is obvious that the standardization condition of vectors x^* and p^* means that:

$$M_c(S) = Xp^*;\ \ M_r(S) = Px^* \tag{43}$$

where X and P are the sums of the components of vectors of standard commodities x^* and standard prices p^*

$$X = \sum_i x_i^*;\ \ P = \sum_i p_i^* \tag{44}$$

Thus, the multipliers of the Sraffian matrix S are:

$$M_c(S) = Xp^*;\ \ M_r(S) = Px^* \tag{45}$$

and $V(A) = PX$. Hence, the multiplier product matrix of the Sraffian matrix coincides with the Sraffian matrix itself:

$$M(S) = S \tag{46}$$

Using this property and applying formulae (42) to the condition $AS=SA=\mu S$ one obtains:

$$M_c(A)S = \mu Xp^*;\ \ SM_r(A) = \mu Px^* \tag{46}$$

This implies that

$$M_c(A)x^* = \mu X;\ \ p^* M_r(A) = \mu P \tag{47}$$

or

$$p^* M(A)x^* = \frac{\mu^2 PX}{V} \tag{48}$$

Further, because the well-known property of the row multipliers:

$$M_c(A) = e - v_a \tag{49}$$

where e = (1,1,..., 1) and vector v_a is the vector of value added, the conditions (43) and (44) imply:

$$\mu p^* = M_r(A)S = eS - v_a S = Xp^* - v_a S$$

or

$$v_a S = (X - \mu)p^* \tag{50}$$

The standardization (11) help find the exact expression for vector of prices p and vector of commodities x that are the solutions of the Sraffian models: conditions $\mu p = pA$ and $\mu x = Ax$ means that p and x are the eigenvectors of A corresponding to the eigenvalue μ. Therefore they are proportional to p^* and x^*:

$$p = \alpha p^*, x = \beta x^*.$$

Introducing them into the standardization condition (11), one obtains $\alpha = \beta = 1 - \mu$, so the solutions for the Sraffa models are:

$$p = (1-\mu)p^*;\ \ x = (1-\mu)x^* \tag{51}$$

Substituting (51) into (5) and (6), the decomposition of prices and commodities in Sraffa-Leontief models into three parts may be obtained, namely, an intermediate inputs part, an interest part and a wage part:

$$p = (1-\mu)\, p^* = (1-\mu)\, p^* A + r\,(1-\mu)\, p^* A + w^*(1-\mu)\, p^*(I-A);$$
$$x = (1-\mu)\, x^* = (1-\mu)\, Ax^* + r\,(1-\mu)\, Ax^* + w^*(1-\mu)(I-A)\, x^* \qquad (52)$$

7. Example.

7.1 Multiplier Product matrix for Chicago 1987 economy.

As an example, let us consider the aggregated 6x6 Chicago 1987 input-Output table derived from a 36-sector version. This table was derived from the Chicago Region Econometric Input-output Model (CREIM), details on which can be found in Israilevich *et al*, (1997). Table 1 presents the sector definition for Chicago table and table 2 presents the aggregated table itself together with the column and row multipliers, the direct inputs backward and forward indices and their rank-size hierarchies.

<<insert tables 1, 2 here>>

Table 2 represents the matrix of direct inputs *A* from the Primary and Dual Sraffa-Leontief models (6) and (7). The column and row multipliers of the matrix *A* defining the Multiplier Product matrix, calculated with the help of the formula (31) (see table 3)

<<insert tables 3, 4 here>>

Rearranging the columns and rows of this matrix according the rank-size hierarchies of column and row sums table 4 will be revealed; this is the numerical presentation of the economic landscape visualized graphically in figure 1.

The spatial presentation of the hierarchies of backward and forward linkages of the matrix *A* is presented in the space of backward and forward linkages in figure 2. Such a representation is useful for the description of the temporal dynamics of these linkages.

7.2 Sraffian matrix for the Chicago economy in 1987.

The column sums from table 2 defined, with the help of (10), the interval that includes the maximal Perronean eigenvalue: $0.2501 < \mu < 0.5261$. The actual value of this maximal eigenvalue is $\mu = 0.3367$, so the maximal rate of profits in Chicago 1987 economy is

$$r_{\max} = \frac{1-\mu}{\mu} = 1.97$$

The left hand and right hand eigenvectors corresponding to this eigenvalue are: standard prices eigenvector:

$$p^* = [0.7398,\ 1.464,\ 0.905,\ 1.219,\ 0.89086,\ 0.9704],\ \ P = 6.1968$$

$$x^* = \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}, X = 1, \quad p^* x^* = 1$$

and standard commodities eigenvector

$$x^* = \begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}, \quad X = 1, \quad p^* x^* = 1$$

The Sraffian matrix of standard commodities, standard prices for Chicago 1987 economy are presented in table 5:

<<insert table 5 here>>

The corresponding economic landscape for this matrix is shown in figure 3

The tables presented above will help us to calculate the decompositions (52) of prices and commodities in Sraffa-Leontief models into three parts: an intermediate inputs part, an interest part and a wage part (*cf.* Steenge, 1997, pp.244-247). This is shown as table A1 in the Appendix.

8. Conclusions and Further Explorations

This paper has revealed an important connection between the Sraffa-Leontief system and some new interpretations afforded by the multiplier product matrix. The properties of the latter matrix offer the potential for comparative analysis across time (for a single economy) or across economies at one point in time. Further considerations could involve the exploration of an expansion of the Sraffa-Leontief models by considering the closed system of a Miyazawa type in which profits and wages are distributed and their impacts on the economy are traced. (*cf.* Miyazawa, 1976). In this sense, the work of Trigg (1999), examining a link between Keynes, Morishima and Miyazawa, provides motivation for the potentially new and innovative insights that can be gained by exploring connections between modeling systems.

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Appendix

Table A1: Decomposition of the Sraffian matrix

$$0.6633\left[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704\right] =$$

$$0.6633\left[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704\right]\begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} +$$

$$+r0.6633\left[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704\right]\begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix} +$$

$$+w*0.6633\left[0.7398, 1.464, 0.905, 1.219, 0.89086, 0.9704\right]\begin{bmatrix} 0.9805 & -0.0025 & -0.0101 & -0.0104 & -0.0011 & -0.0012 \\ -0.0273 & 0.9992 & -0.0049 & -0.0245 & -0.0054 & -0.0129 \\ -0.0867 & -0.3109 & 0.8293 & -0.0889 & -0.0475 & -0.0816 \\ -0.0245 & -0.0381 & -0.0381 & 0.8458 & -0.0600 & -0.0418 \\ -0.0245 & -0.0631 & -0.0385 & -0.0153 & 0.9867 & -0.0150 \\ -0.0676 & -0.1106 & -0.0582 & -0.0776 & -0.1740 & 0.8240 \end{bmatrix};$$

$$0.6633\begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} = 0.6633\begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix}\begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} +$$

$$+r0.6633\begin{bmatrix} 0.0195 & 0.0025 & 0.0101 & 0.0104 & 0.0011 & 0.0012 \\ 0.0273 & 0.0008 & 0.0049 & 0.0245 & 0.0054 & 0.0129 \\ 0.0867 & 0.3109 & 0.1707 & 0.0889 & 0.0475 & 0.0816 \\ 0.0245 & 0.0381 & 0.0381 & 0.1542 & 0.0600 & 0.0418 \\ 0.0245 & 0.0631 & 0.0385 & 0.0153 & 0.0133 & 0.0150 \\ 0.0676 & 0.1106 & 0.0582 & 0.0776 & 0.1740 & 0.1760 \end{bmatrix}\begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix} +$$

$$+w*0.6633\begin{bmatrix} 0.9805 & -0.0025 & -0.0101 & -0.0104 & -0.0011 & -0.0012 \\ -0.0273 & 0.9992 & -0.0049 & -0.0245 & -0.0054 & -0.0129 \\ -0.0867 & -0.3109 & 0.8293 & -0.0889 & -0.0475 & -0.0816 \\ -0.0245 & -0.0381 & -0.0381 & 0.8458 & -0.0600 & -0.0418 \\ -0.0245 & -0.0631 & -0.0385 & -0.0153 & 0.9867 & -0.0150 \\ -0.0676 & -0.1106 & -0.0582 & -0.0776 & -0.1740 & 0.8240 \end{bmatrix}\begin{bmatrix} 0.0192 \\ 0.0341 \\ 0.3568 \\ 0.1875 \\ 0.0746 \\ 0.3308 \end{bmatrix}$$

Table 1. Chicago 1987 Input-Output table sectors definitions

Sectors			
Aggregate	original	*Description*	*SIC codes*
AGM	*1*	Livestock and Other Agricultural Products	01,02
	2	Forestry and Fishery; Agricultural Services	07-09
	3	Mining	10-14
CNS	*4*	Construction	15-17
MNF	*5*	Food and Kindred Products	20
	6	Tobacco Manufactures	21
	7	Textiles and Apparel	22-23
	8	Lumber and wood Products	24
	9	Furniture and Fixtures	25
	10	Paper and Allied Products	26
	11	Printing and Publishing	27
	12	Chemicals and Allied Products	28
	13	Petroleum Refining and Related Industries	29
	14	Rubber and Miscellaneous Plastics Products	30
	15	Leather and Leather Products	31
	16	Stone, Clay, Glass and Concrete Products	32
	17	Primary Metal Industries	33
	18	Fabricated Metal products	34
	19	Machinery, Except Electrical	35
	20	Electrical and Electronic Machinery	36
	21	Transportation Equipment	37
	22	Scientific Instruments, Photographic and Medical Goods	38
	23	Miscellaneous Manufacturing Industries	39
TCG	*24*	Transportation and Warehousing	40-42, 44-47
	25	Communication	48
	26	Electric, Gas and Sanitary Services	49
WRT	*27*	Wholesale and Retail Trade	50-57, 59
SRV	*28*	Finance and Insurance	60-64, 67
	29	Real Estate and Rental	65, 66
	30	Hotels, Personal and Business Services	70-73, 76, 81, 89
	31	Eating and Drinking Places	58
	32	Automobile Repair and Services	75
	33	Amusement and Recreation Services	78,79
	34	Health, Educational and Nonprofit Organizations	80, 82-84, 86
	35	Federal Government Enterprises	
	36	State and Local Government Enterprises	

Table 2. Chicago 1987 direct inputs Input-Output table.

Sectors	*AGM*	*CNS*	*MNF*	*TCG*	*WRT*	*SRV*	*Row sums*	*Forward Linkages DIFL*	*Rank-Size Hierarchy of DIFL*
AGM	0.0195	0.0025	0.0101	0.0104	0.0011	0.0012	0.0447	0.1280	VI
CNS	0.0273	0.0008	0.0049	0.0245	0.0054	0.0129	0.0758	0.2168	V
MNF	0.0867	0.3109	0.1707	0.0889	0.0475	0.0816	0.7863	2.2497	I
TCG	0.0245	0.0381	0.0381	0.1542	0.0600	0.0418	0.3567	1.0205	III
WRT	0.0245	0.0631	0.0385	0.0153	0.0133	0.0150	0.1697	0.4856	IV
SRV	0.0676	0.1106	0.0582	0.0776	0.1740	0.1760	0.6639	1.8995	II
Column sums	0.2501	0.5261	0.3204	0.3708	0.3011	0.3287			
Backward Linkages DIBL	0.7156	1.5051	0.9166	1.0608	0.8615	0.9403			
Rank-Size Hierarchy of DIBL	*VI*	*I*	*IV*	*II*	*V*	*III*			

Table 3. Multiplier Product Matrix for Chicago 1987
[direct inputs input-output table]

Sectors	*AGM*	*CNS*	*MNF*	*TCG*	*WRT*	*SRV*	*Row sums*	*Rank-Size Hierarchy of row sums*
AGM	0.00534	0.01124	0.00684	0.00792	0.00643	0.00702	0.0448	VI
CNS	0.00904	0.01902	0.01158	0.01340	0.01088	0.01188	0.0758	V
MNF	0.09377	0.19725	0.12013	0.13902	0.11289	0.12324	0.7863	I
TCG	0.04254	0.08948	0.05450	0.06307	0.05121	0.05591	0.3567	III
WRT	0.02024	0.04257	0.02593	0.03000	0.02436	0.02660	0.1697	IV
SRV	0.07917	0.16655	0.10143	0.11738	0.09532	0.10406	0.6639	II
Column sums	0.2501	0.5261	0.3204	0.3708	0.3011	0.3287		
Rank-Size Hierarchy of Column sums	*VI*	*I*	*IV*	*II*	*V*	*III*		

Table 4. Economic Landscape for Chicago 1987 Multiplier Product Matrix.

Sectors	*CNS*	*TCG*	*SRV*	*MNF*	*WRT*	*AGM*	Row *sums*	*Rank-Size Hierarchy of row sums*
MNF	0.19725	0.13902	0.12324	0.12013	0.11289	0.09377	0.7863	I
SRV	0.16655	0.11738	0.10406	0.10143	0.09532	0.07917	0.6639	II
TCG	0.08948	0.06307	0.05591	0.05450	0.05121	0.04254	0.3567	III
WRT	0.04257	0.03000	0.02660	0.02593	0.02436	0.02024	0.1697	IV
CNS	0.01902	0.01340	0.01188	0.01158	0.01088	0.00904	0.0758	V
AGM	0.01124	0.00792	0.00702	0.00684	0.00643	0.00534	0.0448	VI
Column sums	0.5261	0.3708	0.3287	0.3204	0.3011	0.2501		
Rank-Size Hierarchy of Column sums	*I*	*II*	*III*	*IV*	*V*	*VI*		

Table 5. Sraffian matrix for Chicago 1987 direct inputs Input-Output table.

Sectors	*AGM*	*CNS*	*MNF*	*TCG*	*WRT*	*SRV*	*Standard Commodities x**	*Row multipliers P x**	*Rank-Size Hierarchy of x**
AGM	0.014204	0.028104	0.017376	0.023405	0.017253	0.018632	0.0192	0.118979	VI
CNS	0.025277	0.049922	0.030861	0.041568	0.030642	0.033091	0.0341	0.211311	V
MNF	0.263961	0.522355	0.322904	0.434939	0.32062	0.346239	0.3568	2.211018	I
TCG	0.136423	0.270108	0.166973	0.224906	0.165792	0.179039	0.1875	1.14331	III
WRT	0.055189	0.109214	0.067513	0.090937	0.067036	0.072392	0.0746	0.462281	IV
SRV	0.244726	0.484291	0.299374	0.403245	0.297257	0.321008	0.3308	2.049901	II
*Column Multipliers X p**	0.7398	0.464	0.905	1.219	0.8986	0.9704			
*Standard Prices p**	0.7398	1.464	0.905	1.219	0.8986	0.9704			
*Rank-Size Hierarchy of p**	*VI*	*I*	*IV*	*II*	*V*	*III*			

Figure 1. Economic Landscape for Chicago 1987 input-Output table

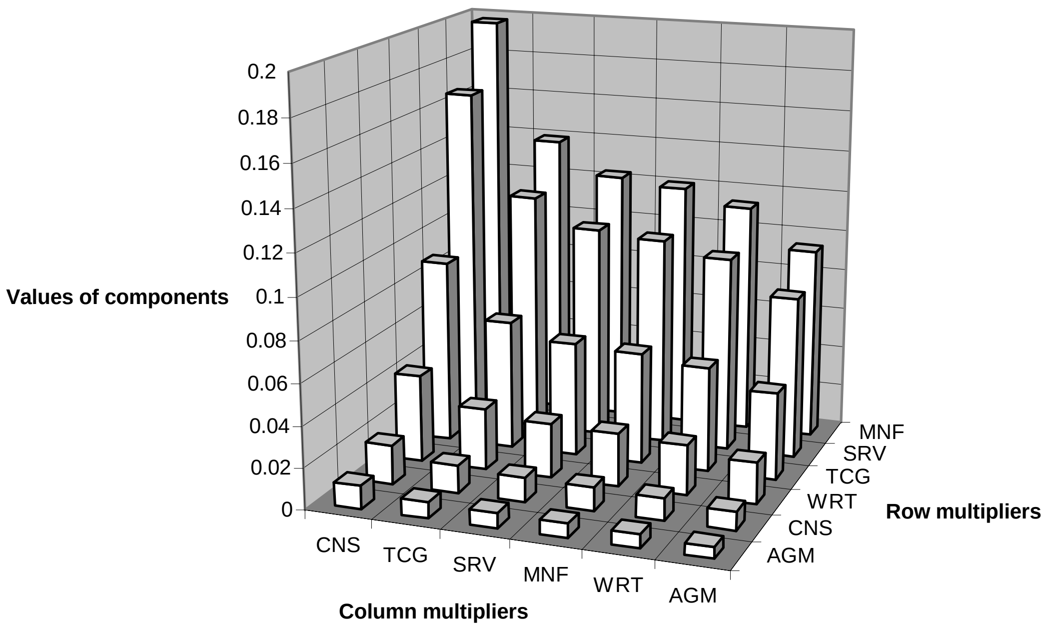

Figure 2. Direct inputs backward and Forward Linkages, Chicago, 1987

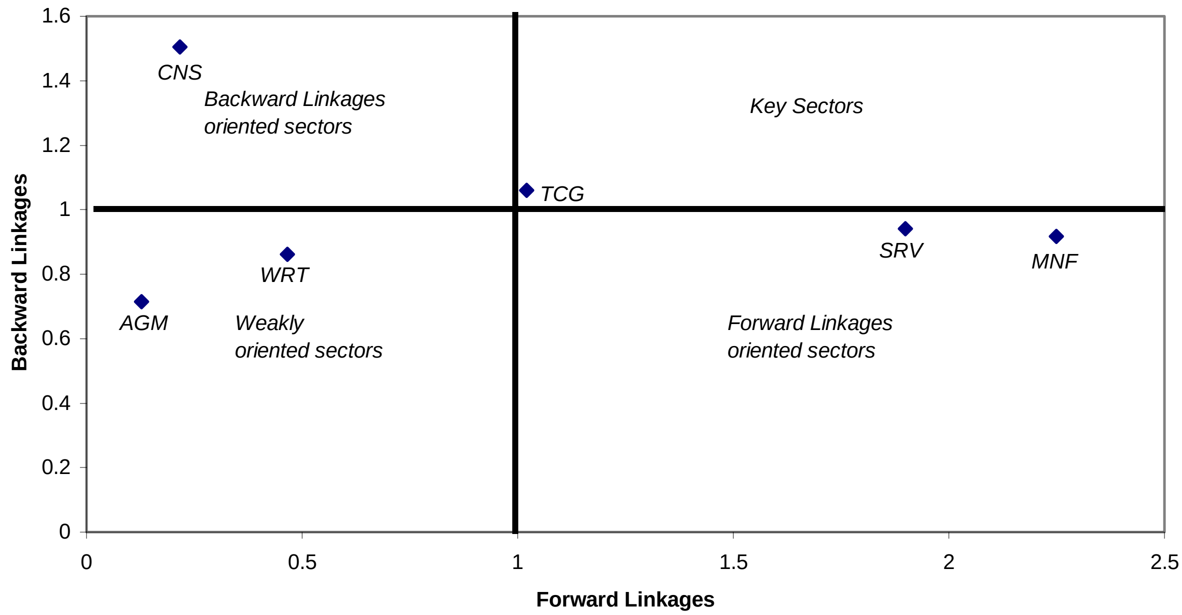

Figure 3 . Economic Landscape of Sraffian matrix